Appendix B: Mathematical Modeling

of the

Mathematics Framework

for California Public Schools: Kindergarten Through Grade Twelve

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Appendix B

Mathematical Modeling

The California Common Core State Standards for Mathematics (CA CCSSM) include mathematical modeling as a Standard for Mathematical Practice (**MP.4,** Model with mathematics), which should be learned by students at every grade level. In higher mathematics, modeling is established as a conceptual category. Additionally, modeling standards are spread throughout other conceptual categories, with a star (★) symbol indicating that they are modeling standards. This appendix serves to clarify the meaning of mathematical modeling and the role of modeling in teaching the CA CCSSM.

What Mathematical Modeling *Is Not*

The terms *model* and *modeling* have several connotations, and although the term *model* has a general definition of "using one thing to represent something else," *mathematical modeling* is a more specific term. Below is a list of some things that do *not* constitute mathematical modeling in the context of the CA CCSSM.

- Telling students, "I do this; now you do the same."
- Using manipulatives to represent mathematical concepts; this might instead be referred to as "using concrete representations."
- Using a graph, equation, or function and calling it a model. True modeling is a process.
- Starting with a real-world situation and solving a math problem. Modeling returns students to a real-world situation and uses mathematics to inform their understanding of the world (i.e., contextualizing and de-contextualizing; see standard **MP.2**).
- Beginning with the mathematics and then moving to the real world. Modeling begins with real-world situations and represents them with mathematics.

What Mathematical Modeling *Is*

Mathematical modeling is the process of using mathematical tools and methods to ask and answer questions about real-world situations (Abrams 2012). Modeling will look different at each grade level, and success with modeling is based on students' mathematical background knowledge as well as their ability to ask modeling questions. However, as discussed below, all mathematical modeling situations share similar features. For example, at a very basic level, grade-four students might be asked to find a way to organize a kitchen schedule to serve a large family holiday meal based on factors such as cooking times, oven availability, cleanup times, equipment use, and so forth (English 2007). The students engage in modeling when they construct their schedule based on non-overlapping time periods for equipment, paying attention to time constraints. When high school students participate in a discussion to evaluate the "efficiency" of the packaging for a 12-pack of juice cans, and then use formulas for area and volume, calculators, dynamic geometry software, and other tools to create their own packaging (making it as efficient as possible), they are also engaged in modeling.

Example of Mathematical Modeling

"Giant's Feet." At Fairytale Town in Sacramento, California, there is a model of the foot of the giant from the story "Jack and the Beanstalk." The foot measures 1.83 meters wide, 4.27 meters long, and 1.27 meters high. If a giant person had feet this large, approximately how tall would he or she be? Explain your solution.

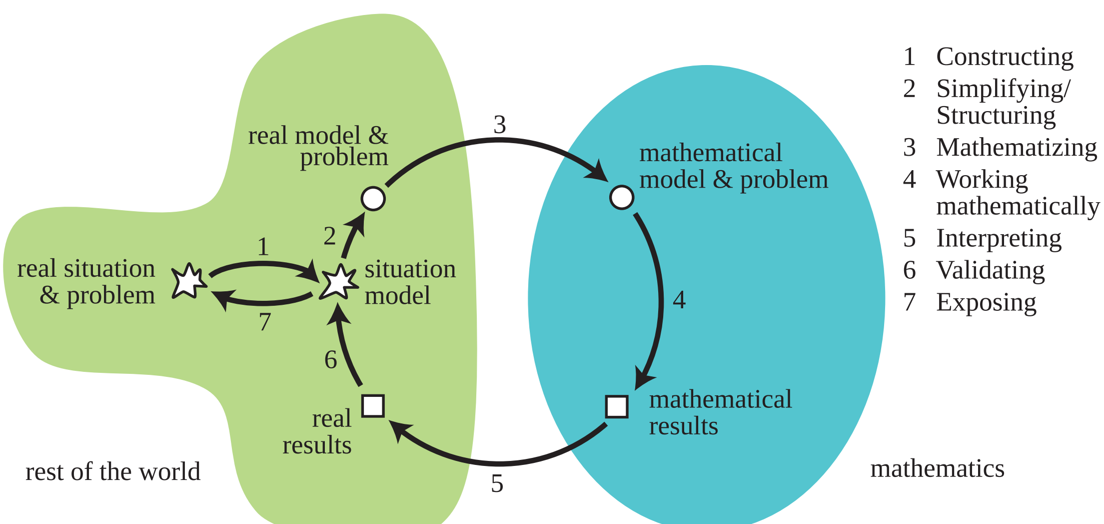
Photo reprinted by permission from Fairytale Town, Sacramento, California.

Mathematical modeling plays a part in many different professions, including engineering, science, economics, and computer science. Professional mathematical modeling often involves looking at a novel real-world problem or situation, asking questions about the situation, creating mathematical representations ("models") that describe the situation (e.g., equations, functions, graphs of data, geometric models, and so on), computing with or extending these representations to learn something new about the situation, and then reflecting on the information found. Students, even those in lower grade levels, can be encouraged to do the same: when presented with a real-world situation, they can ask questions that lead to applying mathematics to new and interesting situations and lead to new mathematical ideas: *How could we measure that? How will that change? Which is more cost-effective, and why?*

Mathematical modeling may be seen as a multi-step process: posing the real-world question, developing a model, solving the problem, checking the reasonableness of the solution, and reporting results or revising the model. These steps all work together, informing one another, until a satisfactory solution is found. Thus, the parameters in a linear model such as $f(x) = 0.8x + 1.5$ may need to be altered to better predict the growth of the supply of a product over time based on initial calculations. Or, a simplification that was made previously in the model formation may need to be revisited to develop a more accurate model.

As shown in figure B-1, Blum and Ferri (2009) offer a schematic that describes a typical modeling process.

Figure B-1. A Typical Modeling Process



Source: Blum and Ferri 2009, 46.

In this cycle, the first step is examining the real world and constructing a problem, typically by asking a question. Second, the important objects or aspects of the problem are identified and, if necessary, simplifications are made (e.g., ignoring that a juice can is not exactly a cylinder). Next, the situation is "mathematized": quantities are identified through measurement, relationships among quantities are described mathematically, or data are collected. This is the step of creating a "mathematical model." Next, the modeler works with his or her model—solving an equation, graphing data, and so forth—and then interprets and validates results in the context of the problem. At this step, the modeler may need to return to his or her model and refine it, creating a looping process. Finally, the results of modeling the problem are disseminated.

The Role of Modeling in Teaching the CA CCSSM

Modeling supports the CA CCSSM goals of preparing all students for college and careers, teaching students that mathematics is a part of their world and can describe the world in surprising ways. Modeling supports the learning of useful skills and procedures, helps develop logical thinking, problem solving, and mathematical habits of mind, and promotes student discourse and reflective discussion. Modeling also allows students to experience the beauty, structure, and usefulness of mathematics.

In contrast with the typical "problem solving" encountered in schools, modeling problems have important mathematical ideas and relationships embedded within the problem context, and students elicit these as they work through the problem (English 2007, 141). In a modeling situation, the exact solution path is often unclear and may involve making assumptions that lead students to use a mathematical skill and reflect on whether they were justified in doing so; this is much different from a word problem in which students are simply required to apply a mathematical skill they have just learned in a new context. Additionally, modeling problems "necessitate the use of important, yet underrepresented, mathematical processes such as constructing, describing, explaining, predicting, and representing, together with organizing, coordinating, quantifying, and transforming data" (English 2007, 141–42). These are some of the same mathematical processes encapsulated in many of the Standards for Mathematical Practice (MP standards). Modeling problems "are also multifaceted and multidisciplinary: students' final products encompass a variety of representational formats, including written text, graphs, tables, diagrams, spreadsheets, and oral reports; the problems also cut across several disciplines including science, history, environmental studies, and literature (English 2007, 141–42).

Current mathematics education literature points to two main uses of modeling in teaching: "modeling as vehicle" and "modeling as content" (see Galbraith 2012).

- **Modeling as vehicle.** According to this perspective, modeling is a way to provide an alternative setting in which students can learn mathematics. This perspective views modeling as a way to motivate and introduce students to new mathematics or to practice and refine their understanding of mathematics they have already learned. When modeling is seen as a vehicle for teaching mathematics, emphasis is not placed on students becoming proficient modelers themselves.
- **Modeling as content.** According to this perspective, modeling is experienced as its own *content*. Specific attention is placed on the development of students' skills as modelers as well as mathematical goals. With modeling as content, mathematical concepts or procedures are not the sole outcome of the modeling activity. As Galbraith (2012) states, "When included as content, mod-

eling sets out to enable students to use their mathematical knowledge to solve real problems, and to continue to develop this ability over time" (Galbraith 2012, 13).

Both of these perspectives on modeling can be included in school mathematics curricula to achieve the complementary goals of having students learn mathematics content and learn how to be modelers. However, the modeling-as-content approach has the additional goal of specifically helping students develop their ability to address problems in their world, which is an important aspect of college and career readiness.

As noted by Burkhardt (2006), people model with mathematics from a very early age: "Children estimate the amount of food in their dish, comparing it with their siblings' portions. They measure their growth by marking their height on a wall. They count to make sure they have a 'fair' number of sweets" (Burkhardt 2006, 181). Zalman Usiskin (2011) notes that the grading system is a stark example of mathematical modeling in many classrooms: "A student obtains a score on a test, typically a single number. This score is on some scale, and that scale is a mathematical model that ostensibly describes how much the student knows . . . the problem to model is that we want to know how much the student knows" (Usiskin 2011, 2). Usiskin also notes that another common example of modeling—determining how big something is—does not appear to be so: "Consider an airplane. We might describe its size by its length, its wingspan, its height off the ground, its weight, the maximum weight it can handle, and the maximum number of passengers it can handle . . . We recognize that [one] cannot describe an airplane's size by a single number" (Usiskin 2011, 3). Still another example of mathematical modeling involves a class of students that will cast votes to elect a new class president. Some voting systems allow each voter to rank the top three candidates and assign different values based on placement (e.g., First = 5 points, Second = 3 points, Third = 1 point). Is this a fair way to determine a winner? These and many other examples show that mathematical modeling occurs from very early on and that modeling questions can arise in many different situations. Thus, there is a unique opportunity in mathematics education to build on this seemingly innate tendency to use modeling to understand the world.

Bringing modeling to the classroom can be a challenging task. The fact that the CA CCSSM focus on *depth* rather than the amount of material covered is an advantage for teachers; having to cover fewer concepts in each grade level or course may allow for more time for modeling experiences that allow students to learn concepts at a deep level. Several challenges to teaching mathematical modeling will arise, not the least of which are understanding the role of the teacher as well as the role of the students, the availability of modeling curriculum, and support for teachers. Each of these issues is discussed in greater detail below, but it is clear that modeling with mathematics will be new to many teachers and students—and therefore it requires care and patience to introduce modeling in a classroom.

Example: Modeling in the Classroom (Grades Four Through Six)

"**Holiday Dinner.**" The three Thompson children—Dan, Sophie, and Eva—want to organize and cook a special holiday dinner for their parents, who will be working at the family store from 7 a.m. until 7 p.m. The children will decorate the house and prepare, cook, and serve the holiday dinner. They know that they need to carefully plan a schedule to get everything done on time. The last time they tried something like this, for their parents' wedding-anniversary dinner, they created an activity list and a schedule for preparing and cooking the meal. Unfortunately, the previous schedule made by the Thompson children did not work very well; they found that they stumbled around the kitchen and wanted to use the same equipment at the same time. They also realized that they had not thought of all the things they needed to include in their schedule.

The children decided on the following menu for their holiday dinner:

- Appetizers (cheese, dip, carrot sticks, and crackers)
- Baked turkey as the main course, served with roasted vegetables and steamed vegetables
- Pavlova,[1] ice-cream, and fresh strawberries for dessert

Dan, Sophie, and Eva know their parents will be home at 7 p.m., and they are all available to begin preparing the dinner at 2:30 p.m. They have four and a half hours to get everything ready. All they need to do is organize a schedule that works better than the wedding-anniversary schedule.

Here are some things the children need to consider:

- How long will it take to cook the turkey?
- What other items can be cooked in the oven with the turkey?
- When should the table be decorated and set?
- When should they make the pavlova, and how long it will take?
- How often do they need to clean in between the cooking?
- How much counter space do they have for food preparation?
- What food needs to be ready first?
- Who will use the equipment, and when?
- How will the tasks be divided among the children?

In the kitchen, there are two counters to work on, a double sink, a microwave oven, and a stove with four top burners and an oven. The oven is large enough to fit the turkey and one other item at the same time.

Dan, Sophie, and Eva need help! They have numerous tasks to complete in order to surprise their parents, and they need a reliable schedule. Students are asked to help the Thompson children in the following ways:

1. Make a preparation and cooking schedule. Chart what each person will do and when, including the use of kitchen equipment.
2. Write an explanation of how you developed the schedule. The children plan to have other surprise celebrations for their parents, and they hope to use your explanation as a guide for making future schedules.

Adapted from English 2007.

1. Pavlova is a dessert consisting of a meringue cake and shell usually topped with whipped cream and fruit.

The Role of the Teacher

The image of students working feverishly in a classroom to solve a real-world problem that resulted from a question they asked paints a different picture of the role of the teacher. When teaching modeling, the teacher is seen as a guide or facilitator who allows students to follow a solution path that they have come up with, making suggestions and asking questions when necessary. Teachers in the modeling classroom are aware of suitable contexts so that their students have an entry point and can ask appropriate questions to attempt to solve the problem. When using modeling to teach certain mathematical concepts, teachers guide the class discussion toward their instructional goal. Teachers in a modeling classroom move away from a role of manager, explainer, and task setter and toward a role of counselor, fellow mathematician, and resource (Burkhardt 2006, 188).

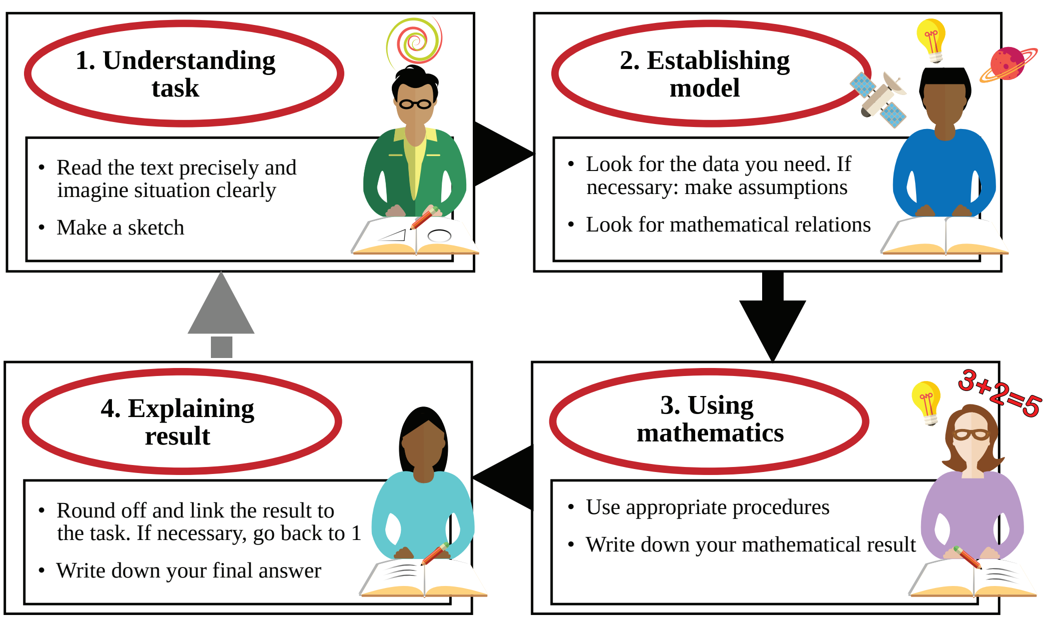
Teachers who are new to modeling may have difficulty allowing their students to grapple with difficult mathematical situations. Modeling involves problem solving, and, as Abrams (2001) states, "Problem solving involves being stuck. If a task does not puzzle us at all, then it is not a problem. It is merely an exercise" (Abrams 2001, 20). Teachers need to remember that learning occurs as the result of struggling with difficult concepts, and thus a certain amount of productive struggle is necessary and desirable.

Blum and Ferri (2009) posit some general implications for teaching modeling based on empirical findings. They note that teachers:

- must provide appropriate modeling tasks for students, and a balance between maximum student independence and minimal teacher guidance should be found;
- should be familiar enough with assigned tasks so that they can support students' individual modeling routes and encourage multiple solutions;
- must be aware of different means of strategic intervention during modeling activities;
- must be aware of ways to support student strategies for solving problems.

As shown in figure B-2, Blum and Ferri (2009) suggest a four-step schematic—simplified from the seven-step process shown in figure B-1—for guiding students' strategies.

Figure B-2. Four Steps for Solving a Modeling Task



Source: Blum and Ferri 2009.

The Role of Students

The transition to modeling in the classroom may prove difficult for students as well as teachers. It is no secret that many mathematics lessons involve a teacher explaining and demonstrating steps while students imitate the teacher. As Burkhardt (2006) notes:

> Most school mathematics curricula are fundamentally imitative—students are only asked to tackle tasks that are closely similar to those they have been shown exactly how to do. This is no preparation for practical problem solving or, indeed, non-routine problem solving in pure mathematics or any other field; in life and work, you meet new situations so you need to learn how to handle problems that are *not* just like those you have tackled before. (Burkhardt 2006, 182)

The transition from passive learner to active learner will pose a major challenge to students who are accustomed to simply mimicking their teacher's actions. However, this transition is empowering for students; they become effective problem solvers when they combine reasoning and persistence to solve problems where the outcome is meaningful to them.

Teachers can help facilitate this transition for students by starting with manageable modeling situations and gradually increasing the complexity of tasks. Students need to learn that in modeling situations, the teacher is a *resource* and not simply a person who provides answers; the students are responsible for doing the hard work. Through entry-level modeling tasks, students can learn to be investigators, managers, and explainers, and they become responsible for their reasoning and its correctness. Eventually, students can pose their own questions and fully carry out the modeling process. Abrams (2012) suggests a "Spectrum of Applied Mathematics" that teachers can follow when providing tasks; this spectrum will allow students to ramp up to full modeling (Abrams 2012, 46). The following spectrum is derived from Abrams's work, and it should be viewed in that context: as a spectrum, not a *ladder*, in the sense that teachers can enter the spectrum in various places according to the needs and abilities of their students.

Spectrum of Mathematical Modeling (Examples Suitable for Upper Middle School and High School)
Level 9 (highest level): Students choose the context and the question. They experience the entire modeling process while confronting two or more iterations. The question may be practical or may concern something about which the student is curious.
Level 8: The context is provided by the teacher. Students determine a meaningful question related to the context and use the modeling process to determine an answer. *Example: Presented with a 12-pack of juice cans (or water bottles), what questions could be asked that would lead to a practical solution?*
Level 7: The teacher determines the context and poses the question to be answered. Students determine the relevant variables, make assumptions, and choose to simplify or ignore some of the variables. Students will need to justify their decision when making presentations. *Example: Find a better way to package juice cans.*
Level 6: Same as level 7, with the exception that the teacher guides students through the process of making assumptions and simplifications. Students develop and apply mathematical models and determine the reasonableness of the solutions. *Example: Find a better (more efficient) way to package juice cans. The discussion will determine that "efficient" means the ratio of the space used to the space available in the package. Students and teachers will assume the cans are perfect cylinders, restrict the package to the height of a single can, orient all cans in the same direction, and use a package that is a prism with congruent polygon bases (no shrink-wrap).*
Level 5: The teacher provides a simplified version of a real-life question and context. The problem is rich enough to allow for several solution paths and allow for access to various levels of mathematical background. *Example: Which package uses the highest percentage of space—a rectangular 12-pack, a triangular 10-pack, a trapezoidal 9-pack, or a hexagonal 7-pack? (All are prisms with a height equal to one can or bottle). Students make accurate representations of these packages and may determine the use of space through measurement, algebraic manipulation applied to polygons, or by using geometric sketchpad software.*
Level 4: Students are guided through the solution process that starts with a real-life context and question. The series of questions ensures that students will follow a particular path and use expected mathematics to solve the problem. The reasonableness of the solution is analyzed. *Example: Which is more efficient—the hexagonal 7-pack or the triangular 10-pack? Determine the percentage of space used in a hexagonal 7-pack.*
Level 3: A context and question are given. This is a real-world context with a mathematical focus. *Example: Six cans (circles) are placed together to form a triangle shape. The design engineer needs to find the height of the configuration. Determine the distance from the bottom can to the top can.*
Level 2: The context or real-world nature is incidental to the problem. The problem may even be contrived. *Example: Three circles are placed tangent to one another. Calculate the area bounded by the three circles.*
Level 1: There is no real-world context; the question is purely mathematical. *Example: Calculate the area of a circle with diameter equal to 2.5 inches.*

The Modeling Curriculum

As noted previously, much of the current mathematics curriculum involves students imitating what their teachers show them. Real-world situations are often employed only as exercises for students to practice mathematics they are currently learning, often in the form of word problems. The preceding discussion already pointed out some features of a modeling curriculum, which includes open-ended tasks, complex problems, student independence, and multiple means of sharing results. Abrams (2012) suggests some differences between true mathematical modeling situations and mathematical exercises (Abrams 2012, 40); see table B-1.

Table B-1. Mathematical Modeling Versus Mathematical Exercises

Mathematical Modeling	Mathematical Exercises
Unfamiliar	Familiar
Memorable	Forgettable
Relevant	Irrelevant
Many possible correct answers	One right answer
Lengthy	Brief
Complex	Simple
Students discover processes	Students follow instructions
Open-ended	Closed (goals chosen by teacher)
Cyclic—constant refining	Linear
Does not appear on a particular page	Appear too often, and then not enough

Modeling often involves project-based work, which can take place over days, weeks, or even months. Modeling problems are unfamiliar and original to students; they are memorable because students must take an active role in the learning process. Modeling problems can be whimsical and clever, with the potential of being extended to the real world. Modeling tasks have an inherent relevance, as students clearly see applications to the real world. Tasks are not predetermined and often have messier endings than traditional problems, as students must sometimes decide whether they have enough information to make a decision. Modeling situations offer great opportunities for cross-disciplinary work and may include problems drawn from the sciences that require students to write reports as a summative activity. Finally, real modeling problems do not come with instructions. Students may take a certain solution path only to find that it did not shed much light on the situation, and then realize that they need to start over along a different path.

Teachers should use their newfound knowledge of the features of modeling to scrutinize instructional materials that are aligned with the CA CCSSM. Teachers may find it necessary to supplement their curricula with rich modeling tasks; see the Related Resources at the end of this appendix for ideas.

Teachers can even make use of problems from traditional curricula in modeling; a traditional word problem can often be changed into a modeling task by asking what would happen if something about the original problem were changed. Teachers can use their own experience to experiment with developing their own modeling tasks for students.

When the goal is the learning or application of particular mathematics content, the challenge for teachers and curriculum developers is to select appropriate investigative tasks that use the modeling process. The tasks must involve question formulation based upon authentic, real-world contexts that are likely to introduce, develop, or apply the desired mathematics content. Because rich real-world contexts are often complicated, simplification in the modeling process is a critical step. Care must be taken to avoid contrived or overly simplified problems. As students develop a deeper understanding of the mathematical modeling process, they should be involved more and more in the stages of formulating questions and simplification. In addition students should experience the latter steps in the mathematical modeling process. Real-world questions should lead to real-world solutions. The solutions are examined for reasonableness (e.g., *Does the answer make sense?*) and usefulness (e.g., *Is the solution applicable to the original situation, or is it necessary to revisit and reformulate the model?*).

Supporting Teachers and Students

With regard to mathematical modeling, many teachers will benefit from their own professional learning, just as they would when dealing with any change in instruction. Teachers in a professional learning setting should experience the process of modeling themselves. By practicing modeling, teachers can get a feel for looking at the world through a mathematical lens; they begin to ask questions, notice mathematically interesting situations, and recognize the usefulness of mathematics in the world. Such exposure is certainly the first step for teachers to take to employ modeling in their classrooms.

Teachers also need experience recognizing, creating, and modifying good modeling problems. Consider Usiskin's "reverse given–find" problems (Usiskin 2011). Typically, mathematics word problems are of the "given–find" variety, wherein certain information is known (i.e., *given*) and students are asked to derive some unknown information (e.g., *find* x). For example, given the sides of a triangle, students are asked to find its perimeter; given the side lengths of a rectangle, students find its area; given a polynomial, students find its roots; and so on. Usiskin suggests reversing these questions: A triangle has perimeter 12 units; what are the possible whole-number side lengths of this triangle? A rectangle has an area of 24 square units; how many rectangles with whole-number side lengths have this area? A polynomial has the following roots . . . can you determine the polynomial? (Usiskin 2011, 5). Of course, this is only one way to develop simple, open-ended situations that are mathematical in nature. However, teachers can start in this way and expand to more complex examples and real-world situations.

Enhancing the Modeling Process

The modeling process is enriched by the following elements:

1. *The facilitative skill of the teacher.* The teacher must create a safe, positive environment in which student ideas and questions are honored and constructive feedback is given by the teacher and by other students. Students do the thinking, problem solving, and analyzing.
2. *The content knowledge of the teacher.* The teacher understands the mathematics relevant to the context well enough to guide students through questioning and reflective listening.
3. *Teacher and student access to a variety of representations and mathematical tools.* Examples include manipulatives and technological tools (dynamic geometry software, spreadsheets, Internet resources, graphing calculators, and so on).
4. *Teacher and student understanding of the modeling process.* Teachers and students who have had prior experience with the modeling process have better understanding of the process and the use of models.
5. *Teacher and student understanding of the context.* Background information or experience may be needed and gained through Internet searches, print media, videos, photographs and drawings, samples, field trips, guest speakers, and so forth.
6. *Richness of the problem to invite open-ended investigation.* Some problems invite a variety of viable answers and multiple ways to represent and solve them. Some contrived problems may appear to be "real-world" but are not realistic or cognitively demanding.
7. *The context of the problem.* Selecting real-world problems is important, and real-world problems that tap into student experiences (prior and future) and interests are preferred.

Teachers need to experience modeling firsthand and develop a different set of skills that they may not possess yet. For example, although the ability to conduct student discourse—allowing discussions to unfold in a non-directive but supportive way and allowing students the time to discuss their ideas—is critical for teaching many of the MP standards, it is of crucial importance for teaching modeling. Teachers also must develop knowledge of tasks, knowledge of the steps of the modeling process so as to identify student difficulties, and knowledge of intervention strategies.

Teachers' belief systems concerning the teaching of mathematics also need to undergo a shift. As mentioned previously, the goals of modeling are not strictly mathematical. Modeling can help students better understand the world and help create a more rounded picture of mathematics for them, in addition to teaching mathematical content. In order to have the time and space needed to implement modeling, teachers need to reflect on their instructional goals and must be willing to practice patience and understanding (with students and themselves), because the modeling process represents a rather large instructional shift.

As noted in the chapter on Supporting High-Quality Common Core Mathematics Instruction, administrators must allow teachers the time and space to implement new CA CCSSM teaching strategies in the classroom. Many of the MP standards are encompassed in the modeling process itself (e.g., standards **MP.1**, **MP.2**, **MP.3**, **MP.5**, and **MP.6**), and therefore teaching with modeling, both as vehicle and as content, supports teaching the CA CCSSM. Administrators must be aware of this and should be supportive of teacher efforts to include modeling in their instruction, especially in middle and high school classrooms. Administrators should be aware that classrooms engaged in modeling tasks are noisy and messy, with students often working excitedly in groups while some students choose to work independently. Teachers with backgrounds in engineering or the sciences are a good resource for implementing modeling.

Finally, the role of parents should also be highlighted. Many great ideas for improving mathematics education have been resisted because of the misunderstanding of parents. In the case of mathematical modeling, problems are messier and teachers are not simply showing students what to do and then having them practice. If parents are informed of the goals and process of modeling, then they can better understand their child's response to the question, "What did you do in math class today?" Parents must be included in the support structure if the CA CCSSM are to be successfully implemented.

Modeling in Higher Mathematics

In the CA CCSSM, modeling is considered a conceptual category for higher mathematics. By the time students have gained proficiency with the K–8 standards, their understanding of number and operations, equations, functions, graphing, and geometry is quite solid. Students further develop these ideas in the higher mathematics courses—especially the notion of *function*, which can play an important role in modeling. While the function concept is developed more fully, and as students' repertoire of expressions and equations increases, students are able to work with more challenging modeling situations.

It is notable that this question may be considered "whimsical": it is a fun question that results in a real-world answer, but it may not be important in the grand scheme of things. However, many mathematical discoveries and questions were discovered by beginning with a similarly whimsical problem, and therefore questions such as this should be explored and encouraged.

The first course in higher mathematics (i.e., Mathematics I or Algebra I) is the first place where mathematical modeling is introduced as a conceptual category. Standard **MP.4** (Model with mathematics) should have been a common experience for students in previous grade levels. Students should have had numerous opportunities to apply mathematical models to solve real-world problems. In Mathematics I or Algebra I, explicit attention is given to teaching the *process* of mathematical modeling. Students learn and practice *all* the steps in the process; they come to understand that the modeling process is seldom linear and that it often involves revisiting steps to formulate a model that is both useful and solvable. The model developed must authentically approximate the real-world context and, at the same time, provide access to students in terms of the mathematics needed to understand the situation or answer the question.

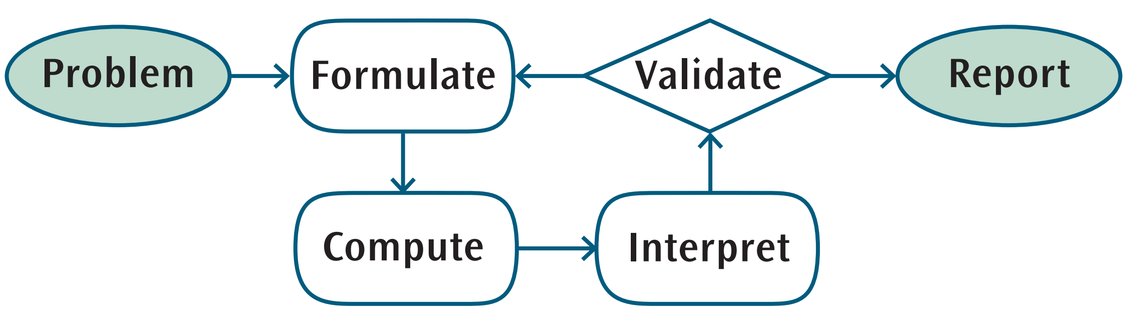
Problems that arise from the real world seldom involve a single content standard. Given enough of their own positive experiences with application problems, teachers and curriculum developers will find that real-life contexts can truly bring out the mathematics that is supposed to be taught in many courses.

Abrams (2012) describes a modeling problem in which students asked themselves, “How can you eat a peanut-butter-cup candy in more than one bite and ensure that each bite has the same ratio of chocolate to peanut butter?” Abrams (2012, 43). After simplifying the peanut-butter cup to two cylinders, with a peanut-butter cylinder embedded in a chocolate cylinder, and simplifying a bite into an arc of a circle intersecting these two cylinders, students tried to discover a formula for the volumes of both chocolate and peanut butter in each bite. The students eventually derived an equation that involves complicated rational expressions contained in square roots, inverse trigonometric functions, and both variables and parameters. They found the ratio in question as a function of the size of the first bite. As Abrams admits, although the problem is not the most important question facing humankind, his students were completely engaged with it and quickly discovered how difficult it was to solve.

It is notable that this question may be considered “whimsical”: it is a fun question that results in a real-world answer, but it may not be important in the grand scheme of things. However, many mathematical discoveries and questions were discovered by beginning with a similarly whimsical problem, and therefore questions such as this should be explored and encouraged.

As shown in figure B-3, the National Governors Association Center for Best Practices, Council of Chief State School Officers (NGA/CCSSO) developed a schematic for modeling at the higher mathematics level.

Figure B-3. The Modeling Cycle



According to the authors:

> The basic modeling cycle is summarized in the diagram. It involves (1) identifying variables in the situation and selecting those that represent essential features, (2) formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables, (3) analyzing and performing operations on these relationships to draw conclusions, (4) interpreting the results of the mathematics in terms of the original situation, (5) validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable, (6) reporting on the conclusions and the reasoning behind them. (NGA/CCSSO 2010a)

The vision of modeling in the higher mathematics standards of the CA CCSSM aligns with much of the discussion in this appendix.

Table B-2 presents some examples of modeling problems suitable for upper middle school and higher mathematics courses.

Table B-2. Modeling in Upper Middle School and Higher Mathematics Courses

Example: Linear Functions. There are numerous real-world contexts that can be modeled with linear functions. Situations involving repetitive addition of a constant amount are plentiful. Common contexts such as comparing cost, revenue, and profit for a business or any context with a fixed and variable component—such as a membership fee combined with a monthly maintenance fee, a down payment followed by monthly payments, or a beginning amount with constant growth, such as simple interest—are opportunities to apply linear models. Students may be asked to determine the feasibility of starting a business—for example, selling hot dogs. A teacher may facilitate a class discussion to identify relevant factors, make assumptions, and gather necessary information. The class might survey the market to determine a reasonable price, such as \$2.25 per hot dog. The cost factors, such as ingredients and paper goods, could be simplified and condensed into a single cost per hot dog, such as \$1.10. The total cost usually includes fixed costs such as rent, licensing or permit requirements, and so forth. Many towns have street fairs or "Market Night" where a space may be rented for a cost (such as \$50 for a four-hour period). Tables, graphs, or equations may be used as models to answer teacher or student questions such as these: *How many hot dogs do I need to sell if I want to make a profit of \$400? How many hot dogs do I have to sell each hour to break even?* Graphing calculators, computer applications, or spreadsheets could also provide powerful models to generalize or extend the investigation.
Example: Exponential Functions. Exponential functions model situations representing a constant multiplier, such as population growth or decay, the elimination of a therapeutic drug in the body, the filtering of harmful pollutants in air or water, compound interest, or cell division. The current population of a town is 18,905. If the population is growing at an average rate of 3 percent each year, when should the population be expected to reach 20,000? What community services might need to increase and thus be reflected in the town's budget? The use of an average rate would be a simplification. Perhaps students would rather investigate a high and low rate to project a range of possible population projections.

Continued on next page

Table B-2 (*continued*)

Example: Juice-Can Packaging. A problem adapted from the National Council of Teachers of Mathematics Illuminations Web site (NCTM Illuminations 2015) involves the packaging of juice cans. [*Note:* water bottles may be substituted for juice cans.] Typically, juice cans are packaged into rectangular prisms in quantities of 6, 12, or 24 cans. The cans are situated in a rectangular array; space between the cans is wasted. Students are challenged to design a new package that will minimize the amount of space that is wasted. Students identify variables and assumptions; the most important aspects are used to focus the problem. Students or the teacher may decide to limit the number of cans to a minimum of 4 and a maximum of 12 with the knowledge that any simplifications may be revisited when the reasonableness of solutions, based on the models, is analyzed. Additional restrictions may include these: the packages are prisms with polygon bases, all cans are situated in the same direction, the cans are perfect cylinders, and the cans are not stacked. The last restriction allows students to simplify the models to two dimensions—circles within polygons—because the dimension of height is held constant for all designs. Students begin the investigation by moving cans around to visualize different arrangements. They represent their designs with careful drawings (circular discs) or by using geometric sketchpad technology. Decisions are made about how to determine whether the wasted space is calculated as an absolute area or volume or a percentage of the available space. Students incorporate the Pythagorean Theorem, equilateral and 30-60-90 right triangles, similar triangles, area of polygons, area of circles, tangents to circles, volume of cylinders, and ratios and proportions.

Creating a Mathematical Modeling Course for High School

A course in mathematical modeling should build upon modeling experiences from previous mathematics courses. The course should allow students to deepen their understanding of the modeling process, apply in new contexts mathematical models they have already learned, and learn new mathematics content to solve unique real-world problems.

Students with a strong background in mathematical modeling should be able to apply mathematics to understand or solve novel problems in career and college settings. A modeling course should allow students to experience all stages of the modeling process, including problem formation; model building that incorporates a variety of mathematical models, skills, and tools for solving the problems; and sufficient analysis to determine if the solution is reasonable or if the model should be revised.

The goal of mathematical modeling is to answer a question, solve a problem, understand a situation, design or improve a product or plan, or make a decision. Mathematical modeling in the school setting includes the additional expectation that students will learn or apply particular mathematical content at a particular grade level. If the learning or application of content standards is the goal, then teachers need to select real-world problems that are likely to have the desired mathematics embedded in them.

Most teachers and students have experienced a single path for learning higher mathematics: it is logical, builds concept upon concept, increases in complexity, and aims to accumulate tools that may be used to solve problems. Rarely, if ever, are students given the opportunity to solve real-world problems in the way that those problems are actually encountered in life. Typically, "application" problems in textbooks are formulated and presented in the form of exercises with the hope that students will buy into the importance of mathematics.

There is another way to learn mathematics that almost no one has experienced in the classroom but most have experienced in everyday life or in a career: Start with a real-world problem or question, apply mathematics already learned to a novel situation, or learn new mathematics that can be applied to solve the problem.

Courses in mathematical modeling may be developed to serve a variety of curricular goals. One course may revisit or build upon modeling standards from previous course work. The emphasis would be on students deepening their understanding and skill by applying previous learning in novel, unique, and unfamiliar situations. Another modeling course may be designed to learn new mathematics (not addressed in previous courses). It should also be noted that real-world problems are not constrained by content standards and often incorporate multiple standards with varied depth. A course in mathematical modeling should extend or supplement—not replace—the integration of modeling into all higher-level mathematics courses and pathways.

Mathematical modeling should be experienced by students and teachers in several ways. Some examples are provided below:

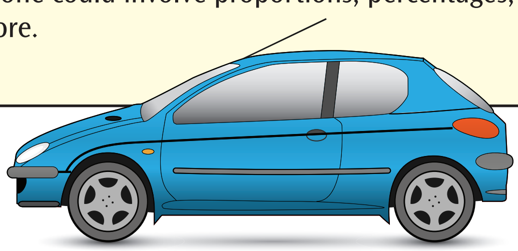
- Short, simple, real-world problems, questions, or "I wonder . . . ?" scenarios that can be solved in just a few minutes by using mathematics
- More complex problems that require additional information acquired through research while also filtering out extraneous information through the process of simplifying and making assumptions. The search for a problem solution would involve more than a day and more than one iteration through the modeling cycle.
- Extensive problems requiring the development of new models and the learning of new mathematics that might constitute an entire unit of study of relevant math content arising from the single real-world scenario. The unit of exploration can extend the mathematics learned to other contexts so that students begin to see the universality of some mathematical models and processes.
- An entire course developed from extensive problem units. The units may or may not be related to one another, but they evolve from real-world situations. The less guidance and scaffolding that is offered by the teacher or the curriculum, the more authentic the modeling experience will be.

Financial literacy is a topic that can find a place in the teaching of the CCSSM in higher mathematics. Topics for a mathematical modeling course could include those dealing with simple cost analysis using linear functions, finding simple and compound interest using exponential functions, finding total cost of payments on a loan, and so forth. Clearly, such topics are relevant to students' future lives and are therefore an important application of modeling.

Example: Owning a Used Car

The teacher poses the following question to a high school class: *How old a car should you buy, and when should you sell it?* The teacher invites students to research several variables online, including options for financing, total cost of the car, depreciation, gas mileage, and the like. Students organize their information and use mathematics to create an argument for why they would buy a car of the model year they have chosen. A modeling situation such as this one could involve proportions, percentages, rates, units, linear functions, exponential functions, and more.

Adapted from Burkhardt 2006, 184.



A Course in Mathematical Modeling Should:

- Be in addition to, not a replacement for, the incorporation of mathematical modeling into the fabric of all higher mathematics courses. The Modeling conceptual category was not intended as a separate course that students may or may not encounter in high school.
- Deepen a student's understanding of, and experience with, all stages of the mathematical modeling process. The course should be about modeling as well as mathematics, and the relationship between the two.
- Allow for sufficient opportunities for students to apply mathematical content they have already learned to unique problems and contexts.
- Challenge and motivate students to recognize the **need** to learn and apply new mathematics and related models. When the models and the mathematics are introduced, students are challenged to find other contexts in which they could apply the same model (or a similar one).
- Progressively allow students more freedom and opportunities to formulate their own questions; develop, apply, and justify their own mathematical models; and analyze and defend their own conclusions through collaboration and dialogue with peers and teachers. (It is challenging for teachers to provide the right balance of freedom and support. Students need to struggle in order for learning to take place, but they should not become so discouraged that they feel like quitting. Teachers need to know which scaffolds to use and should develop open-ended questions to support and sometimes guide students' thinking.)
- Help students and teachers recognize that, to some extent, all people engage in mathematical modeling every day. This can be accomplished by fostering two related dispositions: (1) the ability to look at a life situation and wonder how mathematics might be applied to understand or solve the situation; and (2) the ability to look at a mathematical concept and wonder how it might be applied to life experiences.
- Provide opportunities for students to tackle real-world problems of different complexity. This includes ordinary tasks such as figuring out which coupon to use, which phone plan to choose, or how much of a tip to leave, as well as more complex decisions involving how to prepare for a natural disaster, how to solve a crime, how to rate products, or how to balance the need for increased energy supplies with the need to protect the environment.
- Allow for the learning of mathematical principles, "big ideas," concepts, procedures, standard models, and skills in a meaningful setting, to establish meaning and relevance before teaching mathematics whenever possible.

Sample Topic Areas in an Applied Mathematical Modeling Course

Each starred (★) standard from the higher mathematics conceptual categories of the CA CCSSM could be considered part of a modeling course and may be combined with other higher mathematics standards when creating a course. Table B-3 offers sample topic areas that might be explored in an applied modeling course. All of the starred (★) standards are listed in table B-4 at the end of this appendix.

Table B-3. Sample Topics for a Mathematical Modeling Course

Topic Area	Sample Contexts or Problems	Intended Mathematics Content
Linear Functions—Part 1		
• Making Money • Membership • Choosing Plans	• Simple business models of cost, revenue, and profit are excellent contexts for modeling with linear functions. A real-life business incorporates several variables that will have to be simplified. Sample questions: *How many units of a particular item must I sell to break even or reach a target profit? Which item should I sell if I want to make the largest profit?* • Which plan should I choose? (Consider phone plans, data plans, health clubs, music clubs, membership benefits, and so forth.) • Should I buy or lease a car? • Should I take a job that pays a regular salary, commission, or a blend of the two?	• Linear functions expressed in tabular, graphical, and symbolic forms • Systems of linear functions and equations • Recursive forms with constant addition
Linear Functions—Part 2		
• Line of Best Fit	• Begin with a set of data that are approximately linear and formulate questions. OR • Begin with a real-world question that is likely to have a linear relationship, and then collect the data through simulation, survey, or activity or search for data on the Internet. • For a particular event, how many people should we plan for and/or how much food should we buy? • Based on past trends, how many tickets or programs should we print for this event? • How many schools will we need? How many people might attend? What will my car's value be in seven years? What do we expect to be the total?	• Linear functions derived from approximately linear data; use residuals to determine appropriateness of a model

Table B-3 (*continued*)

Table B-3. Sample Topics for a Mathematical Modeling Course		
Topic Area	**Sample Contexts or Problems**	**Intended Mathematics Content**
Exponential Functions	• Contexts related to growth of populations (people, animal, bacteria, disease) or money (compound interest). Make predictions and/or plans based on anticipated growth of the population or money. • Contexts related to decline or decay of populations (such as half-life of over-the-counter or prescription drugs or depreciation of money) • Contexts related to filtration (such as fans to exhaust particulates from a room or filters that remove pollutants from a water supply) • Whimsical problems that have characteristics that are similar to real-world problems	• Exponential functions expressed in tabular, graphical, and symbolic forms • Equations derived from exponential functions • Decisions to make about limiting the domain to whole numbers, integers, or rational numbers for the base and/or the exponent • Recursive forms involving a constant multiplier • Inverse of exponential function (introduced informally and identified as a logarithm)
Quadratic Functions	• Contexts related to the Pythagorean Theorem and distance. Possibly explore parabolic presence in satellite dishes, telescopes, searchlights, covert listening devices, and solar cookers. • Contexts related to the sum of a series (e.g., carpet rolls and paper rolls) • Contexts related to projectile motion • Contexts related to area • Contexts related to cost, revenue, and profit where price is a linear function	• Quadratic functions expressed in tabular, graphical, and symbolic forms • Equations derived from quadratic functions • Other content that can be connected to geometric contexts
Polynomials	• Problems related to volume (e.g., Produce a box with maximum volume from a flat piece of card stock, but cut out the corners)	• Add, subtract, multiply, and divide polynomials. • Construct polynomials from a real-world situation. • Relate polynomials to geometry.
Absolute Value	• In a town with parallel and perpendicular streets, what is the best location for a new school, hospital, or mall? • Contexts related to tolerance (e.g., factory specifications for a door indicate the door should be 36” wide with a tolerance of 1/32”)	• Absolute value in the real world

Table B-3 (*continued*)

Table B-3. Sample Topics for a Mathematical Modeling Course		
Topic Area	**Sample Contexts or Problems**	**Intended Mathematics Content**
Probabilistic	• Drug testing and determining the cost to test a pool of samples versus testing individual blood samples. For example, if 10 blood samples are pooled and tested as one and the results are negative, then you have saved the cost of testing nine other samples. If the result is positive, then you have to re-test the samples individually or in a smaller group. • Genetic combinations • Fingerprint and DNA testing	• Expected values for false positives and false negatives
Mixed	• How is mathematics used to build the code for representing the movement of objects on a screen or within a video game? • How do blood-spatter patterns help in a crime scene investigation? • Will an asteroid collide with Earth?	• Quadratic functions (for movement of projectiles affected by gravity, such as basketballs) • Parametric equations involving time and location in two or three dimensions • Movements driven or altered by forces and represented by vectors that would then involve trigonometry ratios and possible law of sines and cosines
Polygons	• How do you accurately enlarge or reduce an object? How does scaling affect surface area and weight? • What is the most efficient package design for the packing of cylinders into right prisms with polygonal bases? • Where should sprinklers be placed to optimize water coverage for a lawn or crops? • Any packaging or tiling context using polygon-shaped objects either as the content objects or as the package	• Similarity, scale factors, and dilations • Perimeter and area • Tessellations (rotations, translations, reflections)
Trigonometry • Right Triangles	• How can you estimate the height of a tall object, such as a tower or mountain, when you are prevented from finding the direct distance along the ground? • Using a device to measure angle of inclination or angle of depression, how do you find (indirectly) the height of an object? • How do you render accurately in a drawing or within a video game the height of an object that is tilted at an angle to the viewing plane?	• Right-triangle ratios (tangent, sine, and cosine)

Table B-3 (*continued*)

Table B-3. Sample Topics for a Mathematical Modeling Course		
Topic Area	**Sample Contexts or Problems**	**Intended Mathematics Content**
Circles	• Which pizza size gives consumers the best deal for the money they will spend? • Contexts involving circular motion (including wheels, gears, belt-driven motors, and so forth)	• Area and circumference • Tangents to circles
Volume and Surface Area	• Maximizing the volume of a container while minimizing the surface area (amount of materials needed to make the container) • Pistons and displacement in an internal-combustion engine • Is King Kong possible? How are surface area, weight, and volume affected by enlargement or reduction due to scale factor? • Whimsical: How large is the giant that would fit into the world's largest pair of shoes? • How much liquid would it take to fill the giant cola bottle that is displayed in Las Vegas?	• Prisms, cylinders, cones, spheres • Scale factors (length, area, volume ratios)

Table B-4. Higher Mathematics Modeling Standards in the CA CCSSM	
Number and Quantity	
N-Q.1	Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. ★
N-Q.2	Define appropriate quantities for the purpose of descriptive modeling. ★
N-Q.3	Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. ★
Algebra	
A-SSE.1	Interpret expressions that represent a quantity in terms of its context. ★ a. Interpret parts of an expression, such as terms, factors, and coefficients ★ b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1+r)^n$ *as the product of P and a factor not depending on P.* ★
A-SSE.3	Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. ★ a. Factor a quadratic expression to reveal the zeros of the function it defines. ★ b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines. ★ c. Use the properties of exponents to transform expressions for exponential functions. *For example, the expression* 1.15^t *can be rewritten as* $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.* ★

Table B-4 (*continued*)

Table B-4. Higher Mathematics Modeling Standards in the CA CCSSM	
A-SSE.4	Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.* ★
A-CED.1	Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions. **CA** ★
A-CED.2	Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
A-CED.3	Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★
A-CED.4	Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.* ★
A-REI.11	Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. ★
Functions	
F-IF.4	For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
F-IF.5	Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* ★
F-IF.6	Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★
F-IF.7	Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★ a. Graph linear and quadratic functions and show intercepts, maxima, and minima. ★ b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. ★ c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. ★ d. (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior. ★ e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★
F-IF.10	(+) **Demonstrate an understanding of functions and equations defined parametrically and graph them. CA** ★

Table B-4 (*continued*)

Table B-4. Higher Mathematics Modeling Standards in the CA CCSSM	
F-BF.1	Write a function that describes a relationship between two quantities. ★ a. Determine an explicit expression, a recursive process, or steps for calculation from a context. ★ b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* ★ c. (+) Compose functions. *For example, if* $T(y)$ *is the temperature in the atmosphere as a function of height, and* $h(t)$ *is the height of a weather balloon as a function of time, then* $T(h(t))$ *is the temperature at the location of the weather balloon as a function of time.* ★
F-BF.2	Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms. ★
F-LE.1	Distinguish between situations that can be modeled with linear functions and with exponential functions. ★ a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals. ★ b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another. ★ c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another. ★
F-LE.2	Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table). ★
F-LE.3	Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function. ★
F-LE.4	For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology. ★
F-LE.4.1	**Prove simple laws of logarithms. CA** ★
F-LE.4.2	**Use the definition of logarithms to translate between logarithms in any base. CA** ★
F-LE.4.3	**Understand and use the properties of logarithms to simplify logarithmic numeric expressions and to identify their approximate values. CA** ★
F-LE.5	Interpret the parameters in a linear or exponential function in terms of a context. ★
F-LE.6	**Apply quadratic functions to physical problems, such as the motion of an object under the force of gravity. CA** ★
F-TF.5	Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline. ★
F-TF.7	(+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context. ★
Geometry	
G-SRT.8	Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems. ★
G-GPE.7	Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula. ★
G-GMD.3	Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems. ★

Table B-4 (*continued*)

Table B-4. Higher Mathematics Modeling Standards in the CA CCSSM	
G-GMD.5	**Know that the effect of a scale factor k greater than zero on length, area, and volume is to multiply each by k, k^2, and k^3, respectively; determine length, area and volume measures using scale factors. CA ★**
G-MG.1	Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder). ★
G-MG.2	Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot). ★
G-MG.3	Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios). ★
Statistics and Probability	
S-ID.1	Represent data with plots on the real number line (dot plots, histograms, and box plots). ★
S-ID.2	Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets. ★
S-ID.3	Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers). ★
S-ID.4	Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve. ★
S-ID.5	Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data. ★
S-ID.6	Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. ★ a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. *Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.* ★ b. Informally assess the fit of a function by plotting and analyzing residuals. ★ c. Fit a linear function for a scatter plot that suggests a linear association. ★
S-ID.7	Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data. ★
S-ID.8	Compute (using technology) and interpret the correlation coefficient of a linear fit. ★
S-ID.9	Distinguish between correlation and causation. ★
S-IC.1	Understand statistics as a process for making inferences about population parameters based on a random sample from that population. ★
S-IC.2	Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?* ★
S-IC.3	Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each. ★
S-IC.4	Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling. ★
S-IC.5	Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant. ★

Table B-4 (*continued*)

Table B-4. Higher Mathematics Modeling Standards in the CA CCSSM	
S-IC.6	Evaluate reports based on data. ★
S-CP.1	Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not"). ★
S-CP.2	Understand that two events A and B are independent if the probability of A and B occurring together is the product of their probabilities, and use this characterization to determine if they are independent. ★
S-CP.3	Understand the conditional probability of A given B as $P(A \text{ and } B)/P(B)$, and interpret independence of A and B as saying that the conditional probability of A given B is the same as the probability of A, and the conditional probability of B given A is the same as the probability of B. ★
S-CP.4	Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.* ★
S-CP.5	Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.* ★
S-CP.6	Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A, and interpret the answer in terms of the model. ★
S-CP.7	Apply the Addition Rule, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, and interpret the answer in terms of the model. ★
S-CP.8	(+) Apply the general Multiplication Rule in a uniform probability model, $P(A \text{ and } B) = P(A)P(B\|A) = P(B)P(A\|B)$, and interpret the answer in terms of the model. ★
S-CP.9	(+) Use permutations and combinations to compute probabilities of compound events and solve problems. ★
S-MD.1	(+) Define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions. ★
S-MD.2	(+) Calculate the expected value of a random variable; interpret it as the mean of the probability distribution. ★
S-MD.3	(+) Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value. *For example, find the theoretical probability distribution for the number of correct answers obtained by guessing on all five questions of a multiple-choice test where each question has four choices, and find the expected grade under various grading schemes.* ★
S-MD.4	(+) Develop a probability distribution for a random variable defined for a sample space in which probabilities are assigned empirically; find the expected value. *For example, find a current data distribution on the number of TV sets per household in the United States, and calculate the expected number of sets per household. How many TV sets would you expect to find in 100 randomly selected households?* ★

Table B-4 (*continued*)

Table B-4. Higher Mathematics Modeling Standards in the CA CCSSM	
S-MD.5	(+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values. ★ a. Find the expected payoff for a game of chance. *For example, find the expected winnings from a state lottery ticket or a game at a fast-food restaurant.* ★ b. Evaluate and compare strategies on the basis of expected values. *For example, compare a high-deductible versus a low-deductible automobile insurance policy using various, but reasonable, chances of having a minor or a major accident.* ★
S-MD.6	(+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). ★
S-MD.7	(+) Analyze decisions and strategies using probability concepts (e.g. product testing, medical testing, pulling a hockey goalie at the end of a game). ★

Related Resources

- Consortium for Mathematics and Its Applications (COMAP): http://www.comap.com/ (accessed September 9, 2015)
  Modeling problems are available at ~~http://www.mathmodels.org/problems/~~ [The link is now invalid.] (accessed September 9, 2015).

- Mathematics TEKS Toolkit, Charles A. Dana Center: http://web.archive.org/web/20160918061708/http://www.utdanacenter.org/mathtoolkit/instruction/activities/models.php (accessed April 13, 2017)

- Illuminations, National Council of Teachers of Mathematics (NCTM): http://illuminations.nctm.org/ (accessed September 9, 2015)

- Illustrative Mathematics: https://www.illustrativemathematics.org/ (accessed September 9, 2015)

- Learning and Education in and through Modeling and Applications (LEMA): ~~http://www.lema-project.org/web.lemaproject/web/eu/tout.php~~

- Shell Center for Mathematical Education, Mathematics Assessment Project: http://map.mathshell.org.uk/materials/tasks.php (accessed September 9, 2015)

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